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SOME RESULTS ON GRAPH REPRESENTATIONS AND GRAPH COLORINGS

BY

ARLENE MIA HEISSAN

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE

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DOCTOR OF PHILOSOPHY DISSERTATION

OF

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UNIVERSITY OF RHODE ISLAND

2013

ABSTRACT

A graph $G(V, E)$ is a structure used to model pairwise relations between a set of objects. In this context, a graph is a collection of *vertices* (representing the objects) and a collection of *edges* (representing the relation) that connect pairs of vertices. It is possible to represent a graph using an adjacency matrix, but often this is not the most efficient representation of the relation. In studying graph representation, the object is to capture the structure of the graph more efficiently using a variety of other discrete structures.

This work considers path representations of graphs. Consider a host graph, H. A path representation $[H : r : q]$ of a target graph G is a labeling in which each vertex is assigned a unique path of length r found in H in such a way that if $uv \in E(G)$, then the P_r assigned to u and the P_r assigned to v have at least a P_q in common. This study considers representations in which the host tree is the complete graph on n vertices, $[K_n, r, q]$ which will be referred to as $P_{r,q}$-representations.

This work also considers the area in graph theory known as vertex-coloring, specifically coloring planar graphs, and explores a special class of planar graphs called "coils".

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DEDICATION

To my mother, Dr. Arlene Lynch.

She taught me how to do the most important math of all ... count my blessings.

PREFACE

In mathematics, a *graph* is a data structure that consists of a finite set of of ordered pairs of *vertices* which represent *edges*. Graph theory is the area of mathematics which studies these data structures. An active area of research within graph theory is the study of graph representations.

Different data structures are used to capture the information contained in a graph: adjacency lists, incidence lists, adjacency matrices, incidence matrices. Each has with it a time complexity cost associated with performing various operations on a graph, for example adding or removing a vertex or an edge. Depending on the nature of the graph, certain structures are preferred over others. If the graph is sparse, it would be preferable to use a list. If the graph is dense (the number of edges $|E|$ is close to the square of the number of vertices $|V^2|$), it would be preferable to use a matrix. Many applications work with graphs with special structures. In studying graph representations we attempt to exploit this structure in order to obtain a simpler and more efficient representation which in turn will reduce the costs associated with computing.

A representation of a target graph G consists of three objects, 1) a host set H, 2) an assignment function f, and 3) a conflict rule g. The assignment function assigns a subset of the host set to each vertex of a target graph. The conflict rule compares these assigned subsets to determine whether or not two vertices should be adjacent in G. If, given a host set H and conflict rule g, there is a suitable assignment function such that the graph G is induced by the conflict rule, we say that G is $[H : g]$-representable.

The first part of this dissertation will look at a special type of graph representation, the $[K_n : P_r, P_q]$-representation where the host graph is the complete graph on n vertices, the vertices are labeled with paths of length r, and vertices are adjacent if they have a path of length q in common. We share our results to date and discuss our proposed work moving forward. This line of research provides ample room for future work. In my dissertation I look at some general cases of $P(r, q)$-representations, but spend considerable effort on $P(3, 1)$. The immediate goal is to classify all graphs which are $P(3, 1)$-representable, and eventually look at $P(4, 2)$-representations.

For the second part of my dissertation I look at a special class of graphs called *planar* graphs. Planar graphs are graphs that may be drawn in the plane in such a way that no edges are crossing. Specifically, I look at a sub-class of planar graphs called *coils*, a planar graph whose depth-first search tree is a path. Here, we are not so concerned with representation but move toward producing a short proof that this special class of coils may be colored using four colors. For more than a century there has been interest in proving the Four-Color Theorem (4CT) by means of a short proof. The conjecture was first posed in 1852 by Francis Guthrie and remained open until 1976 when Appel and Haken of the University of Illinois discovered its first proof.

The proof was the subject of much controversy, as it relied on an assumed accuracy of computers to check almost two-thousand cases. This proof was the first of its kind. In the years since, many mathematicians have revisited the 4CT. In 1996, Robertson, Sanders, Seymour, and Thomas published a new computer-assisted proof by analyzing only 633 cases.

In 2004, Werner and Gonthier used the Coq proof assistant, a formal proof management system developed in France, to discover a proof based on the 1996 proof, but with some original content. Although the proofs mentioned are generally accepted within the mathematics community, a short self-contained proof is still desirable. Attempts to find such a proof have resulted in the exploration of interesting generalizations of graph colorings, including 'list colorings' and 'defective colorings'. These variations provide many interesting questions for further research opportunities. In this part of my dissertation I make a conjecture about a lower bound for the number of colorings that exist in coloring a coil using only four colors and outline a proof which is self-contained and used counting techniques.

Each part of my dissertation discusses open problems. There are many interesting questions which remain unanswered in both areas of graph representations and colorings, providing a lifetime worth of research opportunities.

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CHAPTER 1

Graph Representations by Subgraphs

1.1 Introduction

The study of graph representations is an active research area in graph theory. Given a graph $G = (V, E)$, a representation of G is the following collection of objects: (1) a set H, (2) a function $f : V \to \mathscr{P}(H)$, and (3) a function $g : f(V) \times f(V) \to \{0, 1\}$ so that $g\,(f(v_1), f(v_2)) = 1$ iff $(v_1, v_2) \in E$. We call H the *host set*, f the *assignment function*, and g the *conflict rule*. We say that a graph G is representable under a given host set H and conflict rule g if there exists a suitable assignment function f.

Much is known about graph representations when the conflict rule depends on intersection between assigned subsets. Intersection Representations have been well-studied by many authors. (See [1] for a comprehensive list.)

Certain substructures within a graph can make that graph difficult or impossible to represent with certain host sets and conflict rules. One such example is the *line graph*. In 1970, Beineke characterized the set of all such graphs. ([2])

1.2 Graphic models

A subgraph of a graph H is a graph G with $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

We start with a *host* graph H and two subgraphs, a *prototype* R, and a *quota* Q. Throughout we will assume that r is the order of R and q the order of Q. A subgraph of H that is isomorphic to a fixed graph C is a *copy* of C in H. An $(H;R,Q)$-*representation* of a graph G is an assignment $v \to R_v$ of each vertex v to a copy R_v of R such that

$$(*) \quad vw \in E(G) \quad \Longleftrightarrow \quad R_v \cap R_w \text{ contains a copy of } Q.$$

The class of all graphs that have an $(H;R,Q)$-representation is $[H;R,Q]$. If any one of H, R, or Q is replaced by a $*$ in this notation, then that parameter is to be regarded as arbitrary. It is conventional to call the graph being represented the *target*. To help distinguish the two levels of abstraction, it is also conventional to refer to the vertices of the target as *vertices*, but to designate the vertices of the host and its subgraphs as *nodes*. The subgraph R_v is the *representing subgraph* for v. The *universal graph* $\Gamma_{[H;R,Q]}$ for $[H;R,Q]$ has all copies of R as vertices, with adjacency determined as in $(*)$.

Line graphs are a classical example.

Definition 1.2.1 *Given a graph G, its* line graph $L(G)$ *is a graph such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are adjacent in G.*

In 1968, Beineke ([2]) proved the following theorem, characterizing line graphs.

Theorem 1.2.2 *The following statements are equivalent for a graph G.*

1. *G is the derived graph of some graph, that is, G is a line graph.*

2. *The edges of G can be partitioned into complete subgraphs in such a way that no vertex belongs to more than two of the subraphs.*

3. *The graph* $K_{1,3}$ *is not an induced subgraph of* G*; and if* abc *and* bcd *are distinct odd triangles, then* a *and* d *are adjacent.*

4. *None of the following nine graphs is an induced subgraph of* G*.*

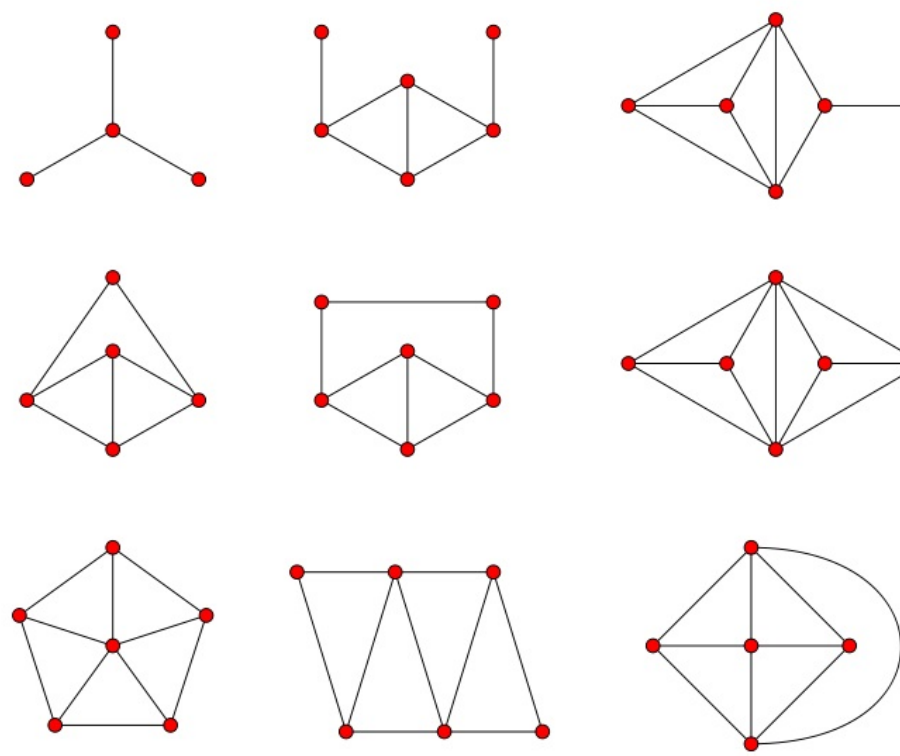

Figure 1. Set of nine forbidden subgraphs.

Here the prototype is the path P_2, the path on two nodes, and the quota is P_1. The class of all line graphs arising from graphs of order $\leq n$ is $[K_n, P_2, P_1]$. The class of *all* line graphs is $[*, P_2, P_1]$.

1.3 Path Representation

We will now look at the structure of graphs in $[*, P_r, P_q]$, where P_k denotes a path on k nodes. For simplicity, we will use the notation $\mathcal{P}(r, q)$ to mean $[*, P_r, P_q]$ and say that a graph has a $P(r, q)$-representation if the graph is in $\mathcal{P}(r, q)$.

Proposition 1.3.1 *$\mathcal{P}(r, q) \subseteq \mathcal{P}(s, q)$ for all $s > r$.*

Proof. Let G be a graph in $\mathcal{P}(r, q)$. Since we are unrestricted on the host graph, attach to the end of each path P_r a vertex not used anywhere in the $P(r, q)$-representation. This yields a $P(r+1, q)$-representation. We repeat this process for a total of $s - r$ times, creating a $P(s, q)$-representation. ■

Proposition 1.3.2 *$\mathcal{P}(r, q) \subseteq \mathcal{P}(r-1, q-1)$ for all $q > 1$.*

Proof. Let G be arbitrary in $\mathcal{P}(r, q)$. Consider the set of all r-length paths used in a $P(r, q)$-representation of G. For each P_r, form the line graph $L(P_r)$, which is itself a P_{r-1}. The collection of these line graphs become the new label set for G, that is, if v_1 had label P^1, its new label is $L(P^1)$. For example, the path 1-2-3-4-5 is now 12-23-34-45=A-B-C-D. It is easily verifiable that vertices in the P_r-labeling share a P_q if and only if the vertices in the P_{r-1}-labeling share a P_{q-1}. Hence, G is in $\mathcal{P}(r-1, q-1)$. ■

The question arises: When is there strict containment? When is there equality?

We spent a great deal of time looking at this question. It lead us to consider the set of line graphs, $\mathcal{L}(G)$ and the set of line graphs of line graphs, $\mathcal{L}^2(G)$. There appears to be some *nesting* properties associated with $\mathcal{L}$ and $\mathcal{P}$ for small values of r and q. We are interested in exploring this idea further. We know that

$\mathcal{P}(2,1) = \mathcal{L}(G)$ (based on their definitions.) We also know from Whitney ([11]) that with one exceptional case (the graphs K_3 and $K_{1,3}$ whose line graphs are both K_3) the structure of a graph G can be completely recovered from its line graph. In other words, G is known if its adjacencies are known. This might prove useful in further characterizing the relationships that exist between line graphs and $P(r,q)$-representations.

Theorem 1.3.3 $\mathcal{P}(4,2) \subsetneq \mathcal{P}(3,1)$.

Proof. Let $H = K_7$ with nodes labeled $1,2,3,4,A,B,C$. Form two partions $X = \{1,2,3,4\}$ and $Y = \{A,B,C\}$. Let the target graph G be the graph labeled with all the P_3's formed by beginning a path in partition X, moving to partition Y, and ending back in partition X. For example, 1-A-2 will be the label of a vertex in G. The resulting graph G will have 18 vertices which is complete with exception of 18 non-edges. Graph G can be partitioned into A, B and C partitions representing all paths that use nodes A, B and C respectively. (See Figure 2.) Notice that the induced subgraph $G_c = G \setminus C$ has a perfect matching of non-edges, as does $G_b = G \setminus B$ and $G_a = G \setminus A$. Also notice that the non-edges in G form three non-adjacent cycles of length 6.

We found a $P(4,2)$ representation of a graph G_{12} on 12 vertices that contains a perfect matching of non-edges (Figure 3). Through exhaustion (see Appendix) we determined that this representation which uses exaclty four nodes from its host graph is the only way (up to isomorphism) to represent G_{12}. Since there are only twelve unique paths using four labels it would be impossible to label three partitions on 18 vertices since the induced graph any two partitions $G_i \cup G_j$ is a copy of G_{12}.

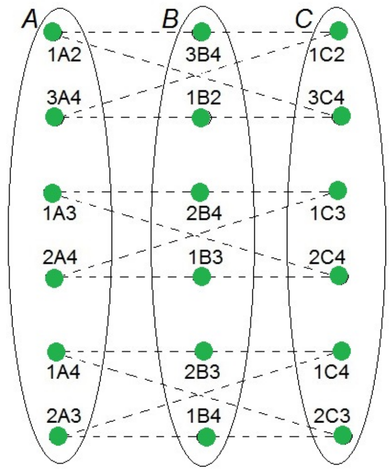

Figure 2. A $P(3,1)$-representation of G.

■

Proposition 1.3.4 $\mathcal{P}(r,q) \subseteq \mathcal{P}(r,q-1)$ *for all* $q > 1$.

Proof. This follows from Propositions 1.3.2 and 1.3.1. ■

Proposition 1.3.5 $\mathcal{P}(n,1) \subseteq \mathcal{P}(kn,k)$.

Proof. Let G have a $P(n,1)$-representation. For each of the n nodes used in the P_n to label the vertex in G, say 1-2-3-...-n, replace with the path 1_1-...-$1_k 2_1$-...-2_k-...-n_1-...-n_k. It is easy to see that this gives to each vertex a P_{kn} label and if two vertices shared a P_1 in the $P(n,1)$-representation, they will share a P_k. Hence, G is in $\mathcal{P}(kn,k)$.

■

The question arises: Is $\mathcal{P}(n,s) \subseteq \mathcal{P}(kn,ks)$ for $s \neq 1$?

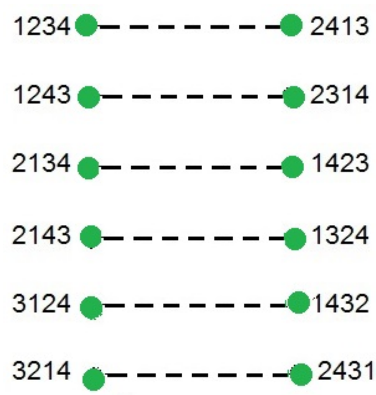

Figure 3. A $P(4, 2)$-representation of G_{12}.

We know that the technique used for proving Proposition 1.3.5 will not work for values of $s > 1$. This does not mean that these graphs are not $P(kn, ks)$-representable, however. There may be a different method for labeling the vertices in such a way as to get a representation. Our initial thought, however, is that the answer to the question is negative.

Theorem 1.3.6 $\mathcal{P}(2, 1) = \mathcal{P}(3, 2)$

Proof. $\mathcal{P}(3, 2) \subseteq \mathcal{P}(2, 1)$ follows from Proposition 1.3.2 so we need only show $\mathcal{P}(2, 1) \subseteq \mathcal{P}(3, 2)$. Assume G has a $P(2, 1)$-representation. Without loss of generality, assume that all labels are ordered chronologically, that is, label 5-4 would be 4-5. Consider a vertex v_1 labeled 1-2. v_1 is adjacent to all vertices labeled 1-x or 2-y where $x \neq 2$ and $y \neq 1$. v_1 is not adjacent to any vertex labeled w-z where $w < z$ and $w, z \notin \{1, 2\}$. Re-label all vertices i-j with the new label i-A-j. v_1 is now labeled with the P_3 1-A-2. All vertices previously labeled 1-x or 2-y are now labeled 1-A-x or 2-A-y, respectively. All vertices previously labeled w-z where $w < z$ and $w, z \notin \{1, 2\}$ are re-labeled w-A-z. v_i is still adjacent to

1-A-x or 2-A-y and not adjacent to vertices labeled w-A-z.

Hence, $\mathcal{P}(2,1) = \mathcal{P}(3,2)$.

■

1.4 Characterization of $P(3,1)$

Theorem 1.4.1 *G has a $P(3,1)$-representation if and only if there exists an edge covering of G into cliques such that each vertex belongs to exactly three cliques and there is no K_4 contained in the intersection of three cliques.*

Proof. Assume that G has a $P(3,1)$-labeling. Without loss of generality, assume the size of the host graph is minimal, that is, all its n nodes are used in the labeling of G. Each node in the host graph H represents a clique in G, that is, $K^1, K^2, ..., K^n$. Since each vertex in G uses exactly three nodes from H in its label, create an edge covering of G where each vertex belongs to exactly three cliques. Since there is a $P(3,1)$-labeling, there does not exist a K_4 contained in the intersection of three cliques in the covering.

Now, assume you can edge cover G into cliques such that each vertex belongs to exactly three cliques and there is no K_4 contained in the intersection of three cliques. Label the cliques $K^1, K^2, ..., K^m$ and use the labels of the cliques to denote the P_3 in H with which to label each vertex. Since there are at most three vertices in the intersection of any of the the cliques, K^a, K^b, K^c, there are three unique paths available to label each vertex: abc, acb, and bac. Hence, G has a $P(3,1)$-labeling. ■

Definition 1.4.2 *Let $\mathcal{F} = \{\mathcal{K}_{1,4}, \mathcal{C}, \mathcal{D}, \mathcal{W}\}$ where*

A	4	7	10	13	19	25	28	37	55	82
B	3	3	3	3	3	2	3	2	2	1
C	3	3	3	2	2	2	1	2	1	1
D	3	2	1	2	1	2	1	1	1	1

Table 1. Values of A, B, C, and D for the set $\mathcal{C}$

- *$\mathcal{K}_{1,4}$ is the set of all graphs containing an induced $K_{1,4}$.*

- *$\mathcal{C}$ is the set of all graphs not in $\mathcal{K}_{1,4}$ that contain a $K = K_A$ in the intersection of three unique K_{A+1}'s, say K^B, K^C, and K^D such that $v_i \in (K^i \setminus K)$ and $N(v_i)$ contains an $\emptyset_i$ for each $i = \{B, C, D\}$. (Refer to Table 1 for values of A, B, C, D, and note that $\emptyset_1$ signifies that the vertex is adjacent to no vertices other than those in K_A. See Figure 4.)*

- *$\mathcal{D}$ is the set of all graphs not in $\mathcal{K}_{1,4}$ that have an induced D_8 (a graph which contains two non-adjacent vertices, x and y that share a neighborhood that is itself a P_6) and are of the following form: there exists a vertex z which is not adjacent to x or y, yet it is adjacent to at least one of the vertices in the P_6=$w_1, w_2, w_3, w_4, w_5, w_6$. Vertex z has the following characteristic: adjacent to the entire P_6; or adjacent to w_2 but not w_3; or adjacent to w_3 but not w_2; or adjacent to w_4 but not w_4; or adjacent to w_5 but not w_4. (See Figure 5.)*

- *$\mathcal{W}$ is the set of all graphs not in $\mathcal{K}_{1,4}$ that are on 7 or 8 vertices and contain a vertex v of degree 6 or 7 respectively, such that for every clique K^a in $N(v)$, the induced subgraph $N(v) \setminus K^a$ yields either an $\emptyset_3$ or a C_5.*

Theorem 1.4.3 *If G is $P(3,1)$-representable, no graph contained in the set $\mathcal{F}$ of forbidden graphs is an induced subgraph of G.*

Proof. $\mathcal{K}_{1,4} \notin \mathcal{P}(3,1)$: Assume G_K is in $\mathcal{K}_{1,4}$ such that $v_1 \in V(G_K)$ and $N(v_1)$ contains an induced $\emptyset_4$. In any edge covering of G_K, v_1 is necessarily contained in four cliques, hence, by Theorem 1.4.1, G_K has no $P(3,1)$-representation.

$\mathcal{C} \notin \mathcal{P}(3,1)$: Assume G_C is in $\mathcal{C}$, that is, G_C contains a $K = K_m$, contained in three unique K_{m+1}'s, say K^1, K^2, and K^3. Let $v_i \in (K^i \setminus K)$. Without loss of generality, label v_1 with the path '1-2-3', v_2 with the path '4-5-6' and v_3 with the path '7-8-9'. Since each vertex in K is adjacent to each v_i it must contain a label from the set $A = \{1,2,3\}$, a label from the set $B = \{4,5,6\}$ and a label from the set $C = \{7,8,9\}$.

The P_3's which use exactly one node from each set represent all possible labels for each vertex in K, assuming that each node in the P_3 is available. With no restrictions, there are $3 \cdot 3 \cdot 3 = 27$ sets of size three which contain exactly one element from the set A, B, and C. Each of these sets of size three can be combined to represent three unique paths from the host graph H. For example, $\{1,4,7\}$ can be used to form the labels 1-4-7, 1-7-4, and 4-1-7, so there are $27 \cdot 3 = 81$ paths available to label the vertices in K, so $m = 82$ is not $P(3,1)$-representable.

Now assume that one of the vertices, v_i, is also adjacent to another vertex, w_i which is not in K. One of the nodes in the path used to label v_i must also be contained in the path used to label w_i, leaving only two nodes available for use in the labeling of the vertices in K. This leaves only $2 \cdot 3 \cdot 3 = 18$ nodes to be used yielding 54 unique paths available to label the vertices in K, so $m = 55$ is not $P(3,1)$-representable.

Now assume that the vertex v_i is adjacent to not only w_i but z_i where w_i and z_i are not adjacent. One of the nodes in the path used to label v_i must be contained in the path used to label w_i, and a different node must be contained in the path used to label z_i, leaving only 1 node available for use in the labeling of the vertices in K. This leaves only $1 \cdot 3 \cdot 3 = 9$ nodes to be used yielding 27 unique paths available to label the vertices in K, so $m = 28$ is not $P(3,1)$-representable.

If we continue in this manner and consider all the different possible adjacencies for each v_i, we see that Table 3.1 gives the various restrictions on m, and the set $\mathcal{C}$ of forbidden subgraphs are not $P(3,1)$-representable.

$\mathcal{D} \notin \mathcal{P}(3,1)$: Assume G_D is in $\mathcal{D}$, that is, it contains an induced D_8 and a vertex z which is not adjacent to vertices x or y, both of which are adjacent to the same $P_6 = (w_1, w_2, w_3, w_4, w_5, w_6)$. Yet, it is adjacent to at least one of the vertices in the P_6. Assume vertex z is adjacent to w_2 yet not adjacent to w_3. If there were a $P(3,1)$-labeling, there would exist an edge covering of G_D in which neither x and y were in 4 cliques. In this case, the edge covering would necessarily contain the following six K_3's: xw_1w_2, xw_3w_4, xw_5w_6, yw_1w_2, yw_3w_4, and yw_5w_6. In this case, w_2 would be in two cliques. Since z is adjacent to w_2, but not w_3, w_2 would be forced to be in a clique containing the edge (w_2, z) and another containing the edge (w_2, w_3). Hence, w_2 would necessarily be contained in four cliques. By Theorem 1.4.1, G_D has no $P(3,1)$-representation. The case is similar for z adjacent to w_3 and not w_2; z adjacent to w_4 and not w_5; and z adjacent to w_5 and not w_4. Now assume that z is adjacent to the entire P_6. Similar to the argument for covering the edges between x and y and the P_6, if there were a $P(3,1)$-labeling, in order to edge cover G_D so that z is not in 4 cliques, the edge covering would necessarily contain the following three additional K_3's: zw_1w_2, zw_3w_4, and zw_5w_6. Hence, each vertex in the P_6 would be contained in three cliques. However, the edges (w_2, w_3) and (w_4, w_5) would still need to be covered, forcing these vertices to be in four cliques. Therefore, by Theorem 1.4.1, G_D has no $P(3,1)$-representation.

$\mathcal{W} \notin \mathcal{P}(3,1)$: Assume G_W is in $\mathcal{W}$ such that $v_1 \in V(G_W)$ and $v_1 \cup N(V_1) = G_W$. Let K be an arbitrary clique in $N(v_1)$ and assume the induced subgraph $N(v_1) \setminus K$ yields an $\emptyset_3$. In any edge covering of G_W, v_1 is contained in at least one clique which covers the edges connecting v_1 to K. In order to cover the edges connecting v_1 to each of the three vertices in the $\emptyset_3$, v_1 must be in three additional cliques, hence, v_1 is necessarily contained in four cliques and by Theorem 1.4.1 has no $P(3,1)$-representation. Now assume that the induced subgraph $N(v_1) \setminus K$ yields a C_5. In this edge covering of G_W, v_1 is contained in at least one clique which covers the edges connecting v_1 to K. In order to cover the edges connecting v_1 to the vertices in the C_5, v_1 must be in three additional cliques, hence v_1 is necessarily contained in four cliques and by Theorem 1.4.1, G_W has no $P(3,1)$-representation.

Every induced subgraph of graph which is $P(3,1)$-representable must also be $P(3,1)$-representable. Hence, if one of the graphs in $\mathcal{F}$ is an induced subgraph of G, then G is not $P(3,1)$-representable. ■

Theorem 1.4.4 *If G is a tree and does not contain an induced $K_{1,4}$, G has a $P(3,1)$-representation.*

Proof. Assume G is a tree and does not contain an induce $K_{1,4}$. Edge cover G using each edge as a clique. Since there are no $K_{1,4}$'s, each vertex is in at most three cliques and by Theorem 1.4.1, G has a $P(3,1)$-representation. ■

Theorem 1.4.5 *If G is a graph on 6 vertices and does not contain an induced $K_{1,4}$, then G is $P(3,1)$-representable.*

Proof. Assume G does not contain an induced $K_{1,4}$. By Proposition 1.3.1, if G has a $P(2,1)$-representation, then it has a $P(3,1)$-representation, so by Theorem 1.2.2 we need only consider the nine Beineke graphs which do not have a $P(2,1)$-representation. Six of the nine graphs are on six vertices, and it is easy to show that these have a $P(3,1)$-representation, so we will focus on the remaining three graphs.

Consider $B_1 = K_{1,3}$ where v_0 is adjacent to v_1, v_2, and v_3, the 'outer vertices'. Without loss of generality, assume v_0 is labeled '1-2-3', v_1 is labeled '1-__-__', v_2 is labeled '2-__-__', and v_3 is labeled '3-__-__'. Let $G = B_1 \cup v_a \cup v_b$ be a graph on 6 vertices. There are $2^9 = 512$ ways to form this union. Many of these have an induced $K_{1,4}$ or do not result in a connected graph. A large number of them are isomorphic to each other. We considered the cases where v_a and v_b are adjacent, and the cases where they are not adjacent. We noted that if either v_a or v_b (or both) is adjacent to v_0, it must also be adjacent to at least one outer vertex in order to avoid a $K_{1,4}$. We also noted that if v_b is not adjacent to v_0 but is adjacent to v_a, v_b may not be adjacent to all three outer vertices or else there is a $K_{1,4}$. In total, there were about 20 cases to check, and in doing so, we verified that if G does not contain a $K_{1,4}$, it is $P(3,1)$-representable.

The remaining two graphs, B_4 and B_9, are on five vertices, so we need only check that the addition of a single vertex, v, which does not result in a $K_{1,4}$ is $P(3,1)$-representable. We give each graph a *relaxed*-$P(2,1)$ representation, that is, one in which duplicate labels are allowed. Hence, each vertex has two labels. It is possible to partition the vertices into three cliques, K^a, K^b and K^c where $|K^i| \leq 3$. Label vertex v 'a-b-c,and give to any vertx in K^i which is adjacent to v the additional label i and if it is not, give to it an arbitrary unused label. Since there are no more than three vertices in a partition, it is possible to re-order the labels to form up to three unique paths, giving each graph a $P(3,1)$-representation. ■

1.5 Future Work

It remains to write out the details of the proof of the following statement: If G is a graph on 7 vertices and does not contain an induced $K_{1,4}$ or an induced subgraph from the forbidden set $\mathcal{W}$, then G is $P(3,1)$-representable.

Outline of proof. Six of the nine graphs are on six vertices. Five of these have a *relaxed*-$P(2,1)$-representation as well as a vertex set that may be partitioned into three cliques of size less than or equal to three. Hence, these graphs are $P(3,1)$-representable. The other graph on six vertices, $B_7 = W_6$ a wheel on six vertices does not have a *relaxed*-$P(2,1)$-representation. The addition a single vertex, v, adjacent only to the vertex of degree 5 is in $\mathcal{W}$ and does not have a $P(3,1)$-representation. In every other case, $B_7 \cup v$ is $P(3,1)$-representable.

Two of the three remaining graphs are on five vertices, B_4 and B_9. We checked all the ways in which two additional vertices could be added. If the newly formed graph does not have a $K_{1,4}$, then it is $P(3,1)$-representable.

The final graph $B_1 = K_{1,3}$ is on four vertices. We checked all the ways to add three vertices to B_1 . If the newly formed graph does not have a $K_{1.4}$ and is not in $\mathcal{W}$, then it is $P(3,1)$-representable.

These cases, however, are important in that we may now focus on graphs that have size at least 8.

Conjecture 1.5.1 *A graph G is $P(3,1)$-representable if and only if no graph contained in the set $\mathcal{F} \cup F'$ of forbidden graphs is an induced subgraph of G.*

We are working on proving that a graph which contains no forbidden subgraphs is $P(3,1)$-representable. We have developed an ordered list of claims that build upon each other. In proving these claims, we hope to identify what the set F' includes or determine that it is empty and out current list of forbidden graphs is complete.

However, there are many questions still open regarding $P(r,q)$-representations. Once we fully classify $P(3,1)$, we will begin to look at $P(4,2)$ and $P(5,3)$ and look for possible generalizations.

List of References

[1] R. Jamison, "Towards a comprehensive theory of conflict-tolerance graphs," *Discrete Applied Mathematics*, 2012.

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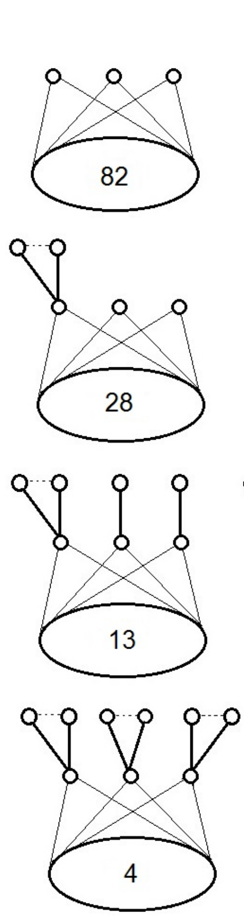

Figure 4. The set $\mathcal{C}$ of forbidden graphs.

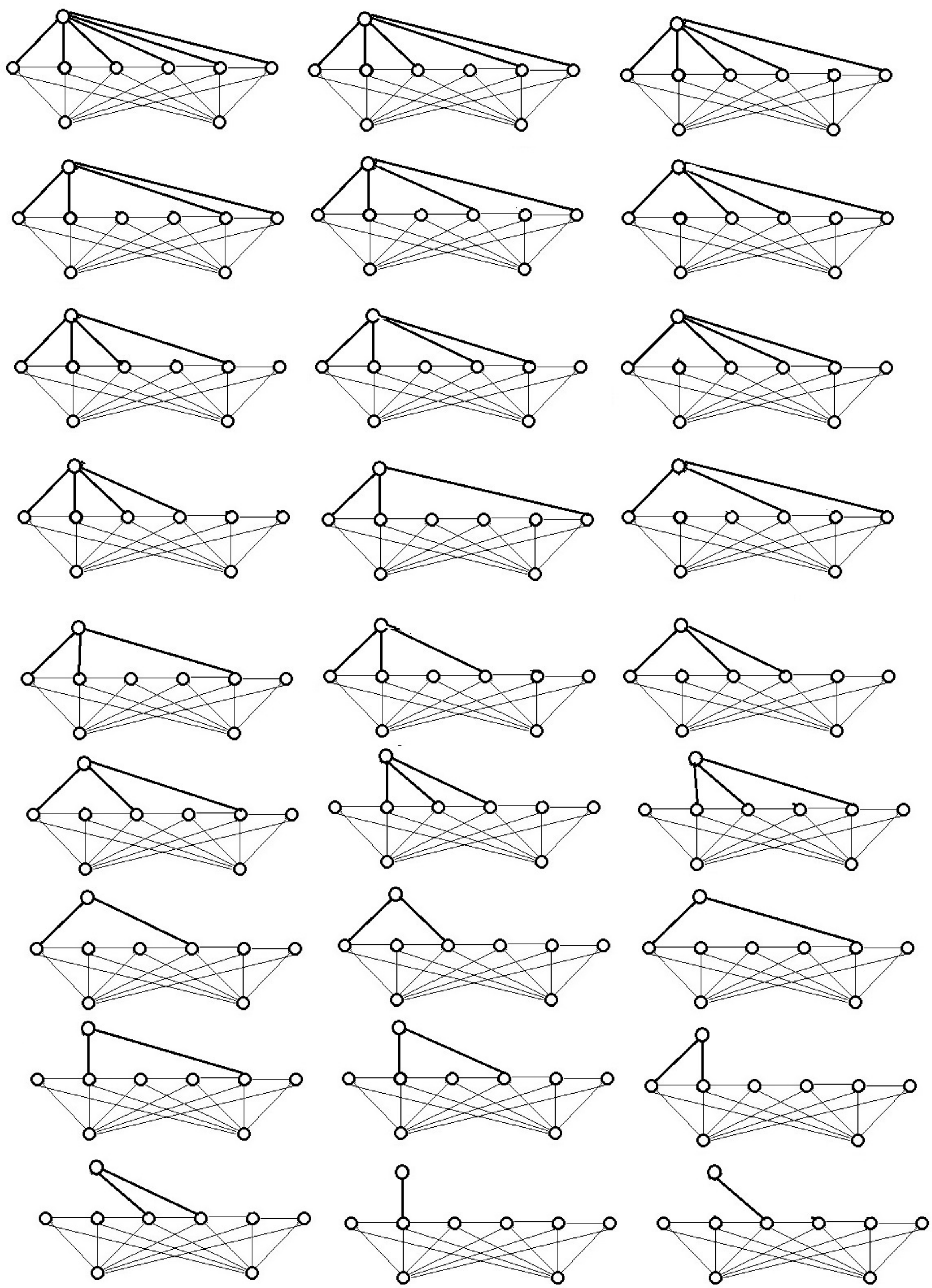

Figure 5. The set $\mathcal{D}$ of forbidden graphs.

CHAPTER 2

Coloring Planar Graphs

2.1 Four-Coloring a Coil

A coil is an inner-triangulated graph whose depth-first search tree $T(G) = (v_1, v_2, \ldots, v_n)$ is a path with the property that for all i, the up-neighborhood $U_i = \{v_j : j < i-1\}$ is a subpath. We conjecture that a coil G is 4-colorable with at least $4 \cdot 3^{n-1} \left(\frac{2}{3}\right)^m \left(\frac{3}{4}\right)^{\beta-1}$ distinct colorings, where m is the number of edges other than path edges, and β is the number of nonempty up-neighborhoods in G.

2.2 Introduction and Definitions

In this paper, we assume that G is a simple, inner-triangulated, and 2-connected plane graph and all 3-cycles, except possibly the outer-circuit, are empty.

For $C' = (w_1, w_2, \ldots, w_k)$, a cycle in G, we denote the vertices embedded in the interior by $I(C)$ and define $G(C')$ to be the induced subgraph on $I(C') \cup C'$.

Let $C = (v_1, v_2, \ldots, v_i^*)$ be the outer-circuit. Induced subgraphs on the neighborhoods of vertices in $I(C)$ form cycles and those in C form paths. Whenever we refer to the neighbors of a vertex as a cycle or a path, we'll assume they are listed in a clockwise direction. Starting with v_1, we produce an ordering of the vertices, $T(G) = (v_1, v_2, \ldots, v_n)$, as they are encountered when forming a depth-first-search tree of G. Thus, the second vertex v_2 is the neighbor of v_1 on the outer-circuit clockwise of v_1 and we proceed in a clockwise direction, using the rule: take the next most clockwise neighbor that hasn't been taken yet. The outer-circuit is the initial subsequence of $T(G)$.

The labeling of the vertices of G suggests an orientation of its edges - (v_s, v_r) is a directed edge if and only $r < s$. Also the children of a vertex $T(G)$ are given an order that indicates the order in which they were chosen in $T(G)$.

Definition 2.2.1 *We consider an interior vertex v_r and its neighborhood $(w_1, w_2, \ldots, w_k)$, where w_1 is its parent in $T(G)$ and w_j is the first child. It is clear that $j > 2$. The neighbors $w_2, w_3, \ldots, w_{j-1}$ of v_r are referred to as its* up-neighbors *and $U_r = (w_2, w_3, \ldots, w_{r-1})$ as its* up-neighborhood. *We define edges of the form $v_s v_r$, where $r < s - 1$, as* crossing edges. *If $T(G)$ is a path and all up-neighborhoods are intervals, we say that $T(G)$ is a* coil. *Thus, each up-neighborhood is an* up-interval.

Let v_α be the vertex with the smallest subscript such that $U_\alpha \neq \emptyset$. Note $\alpha \geq 3$. Let $\beta = n-(\alpha-1)$ be the number of (not necessarily distinct) non-empty up-intervals. We see that m, the number of crossing edges, is at most $2n - 5$.
We denote the rooted full-ternary tree with n levels and root r by $\mathcal{T}$. The coloring ϕ assigns colors $\{1, 2, 3, 4\}$ to the vertices of $\mathcal{T}$ according to the following rule: r gets 1 and the three children of each parent are given distinct colors that are different from the parent's color. We denote by $\mathcal{T} = (T, \phi)$ the tree $\mathcal{T}$ that is colored by ϕ. If a proper coloring of G exists, it is represented by some path in $\mathcal{T}$ from the root to level n. In this paper, we derive a minimum positive bound on the number of such paths that depends on the number of vertices, crossing edges, and nonempty up-intervals in G.

Definition 2.2.2 *Let $U_\alpha = (v_1, v_2, \ldots, v_{\ell_\alpha})$. We let $\mathcal{T}(U_\alpha)$ denote the subtree of $\mathcal{T}$ that consists of all of the intervals from levels 1 through ℓ_α that are colored with at most three colors. We refer to the paths in $\mathcal{T}^*(U_\alpha)$ that extend from the root to its leaves as* interval-paths.

Definition 2.2.3 *For $i \leq n$, let $X_i = G[\{v_i\} \bigcup U_i]$ be the fan associated with v_i. Define X_{n+1} to be the P_3: v_n, v_{n-1}, v_{n-2}, and G_j to be the union $\bigcup_{i=j}^{n+1} X_i$.*

Definition 2.2.4 *The **up-degree** of a vertex v_i is the size of its up-interval; up-deg(v_i) $= |U_i|$.*

2.3 Preliminary Lemmas

Lemma 2.3.1 *A coil G on n vertices, with k crossing edges and one up-interval has at least $C = 3^{n-1}\left(\frac{2}{3}\right)^k$ proper colorings, such that the color of v_1 is 1.*

Proof. Note that $n = k+2$ so that $3^{n-1}\left(\frac{2}{3}\right)^k = 3^{k+1}\left(\frac{2}{3}\right)^k = 3 \cdot 2^k$. G is the wheel W_n on $k+2$ vertices, and it is clear that W_n can be colored as desired. ∎

Lemma 2.3.2 *Let G be a coil on n vertices, with k crossing edges and one up-interval U_n. If $k = 1$, $\mathcal{T}^*(U_n)$ has one node. For $k > 1$, $\mathcal{T}^*(U_n)$ has $3(2^{k-1} - 1)$ nodes at level k. (giving $3(2^{k-1} - 1)$ interval-paths).*

Proof. We are concerned only with the portion of the tree which represents the up-interval U_n, so we consider the subtree $\mathcal{T}^*(U_n)$ of height k. For $k = 2$, there are $3 = 3(2^{2-1} - 1)$ nodes at level k, so the base case holds. By induction, assume the Lemma is true for $n - 1 = k + 1$ and the tree $\mathcal{T}^*(U_{n-1})$ of height $k - 1$ has $3(2^{k-2} - 1)$ nodes at level $k - 1$.

In adding one more vertex to the coil, the up-interval U_n has k vertices. We see $\mathcal{T}^*(U_n)$ as an extension of $\mathcal{T}^*(U_{n-1})$ to the k^{th} level so that each interval-path through the first k levels uses no more than three colors. Each node at level $k-1$ has at least two children, since the $k-1$ interval path from level 1 to level $k-1$ is missing at least one of the 4 colors. Also, there are exactly three nodes that have a third child, since exactly three of the paths are missing two colors (the root node is fixed at color 1). This gives $2(3(2^{k-2} - 1)) + 3 = 3(2^{k-1} - 1)$ nodes at level k as desired. ∎

Lemma 2.3.3 *Let G be a coil on n vertices, with k crossing edges and one up-interval. There exists a corresponding tree $\mathcal{T}(G)$ that contains $C = 3 \cdot 2^k$ full paths from the root to level n, which represent proper colorings of G, such that the color of v_1 is 1.*

Proof. Assume the root node of $\mathcal{T}^*(U_n)$ is colored 1. Consider the rooted subtrees of $\mathcal{T}^*(U_n)$ consisting of the bottom three levels of the tree, that is, the collections of subtrees whose roots are at level k and contains nine leaves at level $k+2$. In each of these subtrees, three of the nine leaves are colored the same color as the root node and the remaining six leaves are colored with the remaining three colors - two each. Note that no matter how large k is, exactly three paths will be missing two colors.

Note that the full paths through $\mathcal{T}^*(U_n)$ represent colorings of G in which $T(G)$ is properly colored and the vertices contained in U_n use at most three colors. We have left only to remove those leaves (corresponding to v_n) whose color is used in the interval-path above it (corresponding to U_n.)

Consider an interval-path through the first k levels. This path uses either 1, 2 or 3 colors.

Case 1 $k = 1$.

Clearly only one color is used as the interval is represented by a single node colored 1, and we keep the leaves which are colored by one of the remaining three unused colors: 2,3 or 4. From the distribution of the nine leaves, we keep 2/3 of the colorings, and we see that $9(2/3) = 3 \cdot 2^1$.

Case 2 $k = 2$.

Clearly each of the interval-paths use only two colors, and we keep the leaves which are colored by one of the remaining two unused colors. From the distribution of the nine leaves of the sub-tree beneath a given interval-path, we keep $4/9$ of the colorings. So, there are 27 full paths, none of which used more than three colors in the first k levels, and $4/9$ of the paths corresponded to proper colorings, and we see that $27(4/9) = 3 \cdot 2^2$.

Case 3 $k \geq 3$.

We consider the subtrees whose roots are on level k and whose leaves are the leaves are on level n. Exactly three of the $3(2^{k-1} - 1)$ interval-paths from level 1 to level k (see Lemma 2.3.2) use only two colors, and we keep the leaves of the subtrees beneath those intervals which are colored by one of the remaining two unused colors. Similar to case 2, we keep $4/9$ of the colorings in these three subtrees. The rest of the interval-paths use three colors, and we keep the leaves which are colored the same as the remaining unused color.

In each subtree, one of the three colors in level $k+1$ is the missing color, therefore only two of the nine nodes on level $k+1$ can be colored with the missing color, that is, we keep $2/9$ of the colorings in these subtrees. This gives

$$3(2^{k-1} - 1) \cdot 3^2 \cdot \left(\frac{3(4/9) + (3(2^{k-1} - 1) - 3)(2/9)}{3(2^{k-1} - 1)} \right) = 3 \cdot 2^k$$

colorings, as desired. ■

Definition 2.3.4 *The color of the root node of $\mathcal{T}^*(U_n)$ is called the **primary** color and the remaining three colors are the **secondary** colors. A node that has the primary color is called a primary node and likewise, a node that has the secondary color is called a secondary node.*

Lemma 2.3.5 *Let G be a coil on n vertices, with k crossing edges and one up-interval. If $k > 1$, $\mathcal{T}^*(U_n)$ has the following distribution of nodes at level k.*

- *For k even, there are $P_k = 2^{k-1} - 2$ primary nodes and $S_k = 2^k - 1$ secondary nodes, with $(2^k - 1)/3$ of each secondary node.*

- *For k odd, there are $P_k = 2^{k-1} - 1$ primary nodes and $S_k = 2^k - 2$ secondary nodes, with $(2^k - 2)/3$ of each secondary node.*

Furthermore, the secondary nodes are equally distributed among the three secondary colors.

Proof. Again, we are concerned only with the portion of the tree which represents the up-interval, so we consider the subtree $\mathcal{T}^*(U_n)$ of height k. It is clear the base case holds for $k = 2$ and $k = 3$.

From Lemma 2.3.2, we know that there are $t_{k-1} = 3(2^{k-2} - 1)$ nodes at level $k-1$. This total can be broken into primary nodes, P_{k-1} and secondary nodes S_{k-1} - with the same number of each secondary node. Then $t_{k-1} = P_{k-1} + S_{k-1}$. From Lemma 2.3.2, $t_k = 2t_{k-1} + 3 = 2(P_k + S_k) + 3$. Since the root node is primary, every path through $t^*(U_n)$ contains primary nodes. At the level $k-1$, any secondary will be extended to primary nodes at the k level, and no primary node at level $k-1$ will be extended to the primary color. So, $P_k = S_{k-1}$. Since nodes are either primary or secondary, we know that $S_k = t_k - P_k = 2(P_{k-1} + S_{k-1}) + 3 - P_k = 2P_{k-1} + 2S_{k-1} + 3 - S_{k-1} = 2P_{k-1} + S_{k-1} + 3$. Using the initial values $P_1 = 1, P_2 = 0, P_3 = 3, S_1 = 0, S_2 = 3, S_3 = 6$, solving the difference equation yields the desired results.

By the symmetry of the tree, the distribution of the secondary nodes must be equal. ■

Lemma 2.3.6 *Of all the colorings in $\mathcal{T}(G)$, which result from Lemma 2.3.3, there exists a collection $\mathcal{S}$ of size at least 3/4 of these colorings where each color class* *at level $n-1$ other than the primary color is of equal size and holds at least 2/9 of the colorings in this collection.*

Proof. Again, for the sake of simplicity, we assume the root node is colored 1, that is, 1 is the primary color of $\mathcal{T}^*(U_n)$ and 2,3 and 4 are the *secondary* colors. Let the notation $\overline{s}$ indicate the form of an interval that does not use the secondary color s. Note that an interval may be of the form $\overline{s_i}$, $\overline{s_i s_j}$, or $\overline{s_i s_j s_k}$.
Let the notation (C_1, C_2, C_3, C_4) indicate the distribution of the colorings which survive after Lemma 2.3.3. C_i gives the number of colorings that have the color i at the $n-1$ level. Given an interval-path, let ϕ denote the color of the bottom node.

Case 1 $k = 1$.

There is only interval path, the single node colored 1 and $\phi = 1$. This interval is $\overline{234}$: $\overline{2}$ yields (0,0,1,1), $\overline{3}$ yields (0,1,0,1), and $\overline{4}$ yields (0,1,1,0). This leaves a distribution of (0,2,2,2) at the $n-1$ level. So, the sizes of the secondary color classes at the $n-1$ level are equal and hold at least 2/9 (in this case 1/3) of the colorings.

Case 2 $k = 2$.

There are three interval paths, all starting with the primary color 1 ending in one of each $\phi = 2, 3$, and 4. These paths are colored $\overline{34}$, $\overline{24}$, and $\overline{23}$ respectively. From $\phi = 2$, colored $\overline{34}$: $\overline{3}$ yields (1,0,0,1) and $\overline{4}$ yields (1,0,1,0). From $\phi = 3$, colored $\overline{24}$: $\overline{2}$ yields (1,0,0,1) and $\overline{4}$ yields (1,1,0,0). From $\phi = 4$, colored $\overline{23}$: $\overline{2}$ yields (1,0,1,0) and $\overline{3}$ yields (1,1,0,0). This leaves a distribution of (6,2,2,2) at the $n-1$ level. Consider a subcollection of size 3/4 of these twelve colorings, that is, nine colorings with distribution (3,2,2,2). The sizes of the secondary color classes at the $n-1$ level are equal and hold at least 2/9 (in this case exactly 2/9) of the colorings of this collection.

Case 3 *$k \geq 3$.*

Case 3.1 *k is odd.*

There are $2^{k-1}-1$ interval-paths beginning with the primary color 1 and also ending in the primary color, that is, $\phi = 1$. Three of these are two-colored intervals.
The path 1-2-1-... is $\overline{34}$: $\overline{3}$ yields (0,1,0,1), $\overline{4}$ yields (0,1,1,0).
The path 1-3-1-... is $\overline{24}$: $\overline{2}$ yields (0,0,1,1), $\overline{4}$ yields (0,1,1,0).
The path 1-4-1-... is $\overline{23}$: $\overline{2}$ yields (0,0,1,1), $\overline{3}$ yields (0,1,0,1).
These three intervals yield 2(0,2,2,2).
Of the remaining interval-paths that end in 1, one-third are $\overline{2}$, each yielding (0,0,1,1); one-third are $\overline{3}$, each yielding (0,1,0,1); and one-third are $\overline{4}$, each yielding (0,1,1,0). These interval paths yield $\frac{2^{k-1}-4}{3}$(0,2,2,2).
From the trees hanging beneath these interval-paths (ending in the primary color 1), we keep a combined distribution of (0,x,x,x) where $x = \frac{2^k+4}{3}$.
There are $2^k - 2$ interval-paths beginning with the primary color 1, ending in a secondary color, 2,3, or 4, that is $\phi = 2, 3$, or 4. In one-third of these paths, $\phi = 2$: half of those are $\overline{3}$ each yielding (1,0,0,1), half are $\overline{4}$ each yielding (1,0,1,0). In one-third of these paths, $\phi = 3$: half of those are $\overline{2}$ each yielding (1,0,0,1), half are $\overline{4}$ each yielding (1,1,0,0). In one-third of these paths, $\phi = 4$: halfof those are $\overline{2}$ each yielding (1,0,1,0), half are $\overline{3}$ each yielding (1,1,0,0). These interval-paths yield $\frac{2^{k-1}-1}{3}$(6,2,2,2).
From the trees hanging beneath these interval-paths ending in the secondary colors 2,3 and 4, we keep a distribution of (3z,z,z,z) where $z = \frac{2^k-2}{3}$.
This gives a total of $6z+3x$ colorings with the distribution ($3z$, $z+x$, $z+x$, $z+x$) at the $n-1$ level. The sizes of the secondary color classes at the $n-1$ level are

equal. Since $x > z$, $z + x > 2z$ and therefore, $z + x$ is clearly at least 2/9 of the colorings.

Case 3.2 *k is even.*

There are $2^{k-1}-2$ interval paths beginning with the primary color 1 and also ending in the primary color 1, that is, $\phi = 1$. One-third are $\overline{2}$ each yielding (0,0,1,1), one-third $\overline{3}$ each yielding (0,1,0,1), and one-third $\overline{4}$ each yielding (0,1,1,0). Combined, these yield $\frac{2^{k-1}-2}{3}$(0,2,2,2). Notice, none can be two-colored.

From the trees hanging beneath these interval paths ending is the primary color 1, we keep a weighted distribution of $(0,x,x,x)$ where $x = \frac{2^k-4}{3}$.

There are $2^k - 1$ interval-paths beginning with the primary color 1, ending in a secondary color 2, 3 or 4, that is, $\phi = 2, 3,$ or 4. Three of these are two-colored intervals.

The path 1-2-1-2-... in which $\phi = 2$ is $\overline{34}$: $\overline{3}$ yields (1,0,0,1), $\overline{4}$ yields (1,0,1,0).

The path 1-3-1-3-... in which $\phi = 3$ is $\overline{24}$: $\overline{2}$ yields (1,0,0,1), $\overline{4}$ yields (1,1,0,0).

The path 1-4-1-4-... in which $\phi = 4$ is $\overline{23}$: $\overline{2}$ yields (1,0,1,0), $\overline{3}$ yields (1,1,0,0).

These three intervals yield (6,2,2,2).

Of the remaining interval-paths that end in a secondary number, one-third end in each 2,3, and 4. In one-third of these paths, $\phi = 2$: half are $\overline{3}$ each yielding (1,0,0,1), half are $\overline{4}$ each yielding (1,0,1,0). In one-third of these paths, $\phi = 3$: half are $\overline{2}$ each yielding (1,0,0,1), half are $\overline{4}$ each yielding (1,1,0,0). In one-third of these paths, $\phi = 4$: half are $\overline{2}$ each yielding (1,0,1,0), half are $\overline{3}$ each yielding (1,1,0,0). Combined, these yield $\frac{2^{k-1}-2}{3}$(6,2,2,2).

From the trees hanging beneath these interval paths ending in a secondary number, we keep a combined weighted distribution of $(3z,z,z,z)$ where $z = \frac{2^k+2}{3}$.

This gives a total of $6z+3x$ colorings with the distribution $(3z,\ z+x,\ z+x,\ z+x)$ at the $n-1$ level.

Leaving out 3 colorings that use the primary color in level $n-1$, we obtain a collection of size $6z+3x-3$ with distribution $(3z-3, z+x, z+x, z+x)$. We see that the size of this subcollection is larger than 3/4 of $6z+3x$ and that the sizes of the secondary color classes at the $n-1$ level are equal and hold 2/9 of the colorings in this collection.

■

Lemma 2.3.7 *Let A and B be non-negative integers, R be the residue class modulo 4 of $3(A+3B)$, and r the residue class modulo 2 of B. If $A+\frac{R}{3}+\frac{2r}{3}\leq 3B$, then there exists an integer C of size at least $\frac{3}{4}(A+3B)$ such that B is at least $\frac{2}{9}C$.*

Proof. If $B\geq \frac{2}{3}A$, we let $C=A+3B$. Since

$$\frac{2}{9}C=\frac{2}{9}(A+3B)\leq\frac{2}{9}(\frac{3}{2}B+3B)=B,$$

we are done.

Assume $B<\frac{2}{3}A$. Let $C=\lfloor\frac{9}{2}B\rfloor=\frac{9}{2}B-\frac{r}{2}$. (Recall that we are assuming $A+\frac{R}{3}+\frac{2r}{3}\leq 3B$ from the statement of the Lemma.) Thus,

$$\begin{aligned}
C &= \frac{18B}{4}-\frac{r}{2}=\frac{9B+9B}{4}-\frac{r}{2}\\
&= \frac{3(3B)+9B}{4}-\frac{r}{2}\geq\frac{3(A+\frac{R}{3}+\frac{2r}{3})+9B}{4}-\frac{r}{2}\\
&= \frac{3A+R+2r+9B}{4}-\frac{r}{2}=\frac{3+R+2r+3(3B)}{4}-\frac{r}{2}\\
&= \frac{3}{4}(A+3B)+\frac{R}{4}+\frac{r}{2}-\frac{r}{2}=\frac{3}{4}(A+3B)+\frac{R}{4}\\
&= \lceil\frac{3}{4}(A+3B)\rceil.
\end{aligned}$$

And, since $C=\frac{9}{2}B-\frac{r}{2}$, we know $\frac{2}{9}C=B-\frac{r}{9}\leq B$. ■

2.4 Main Conjecture

Conjecture 2.4.1 *Let G be a coil on n vertices with m crossing edges and β nonempty up-intervals. Denote by α the least subscript of a vertex that has a* *nonempty up-interval. There is a corresponding colored tree $\mathcal{T}(G)$ that contains at least $C = 3^{n-1}\left(\frac{2}{3}\right)^m\left(\frac{3}{4}\right)^{\beta-1}$ full-paths, each of which represents a proper coloring of G. Also, there exists a sub-collection $\mathcal{S}$ of these paths of size at least $3/4 \cdot C$, where each color class at level $\alpha-1$ other than the primary color P_α holds an equal number s of colorings, where $s \geq 2/9 \cdot |\mathcal{S}|$.*

Corollary 2.4.2 *Based on this conjecture, all coils are 4-colorable.*

Idea for Proof. We make great strides for proving this conjecture with a number of important lemmas and observations.

We will prove the case for $P_\alpha = 1$. The proof is by induction on β.

Lemmas 2.3.3 and 2.3.6 serve as the base case. Thus let $\mathcal{T}(G_n) = T(G)$ from Lemma 2.3.3 and $\mathcal{S}_n = \mathcal{S}$ from Lemma 2.3.6. Assume $\beta > 1$. Set u to be the up-degree of v_α. Then $n-u-1$ is the order and $m-u$ is the number of crossing edges of $G_{\alpha+1}$. By induction, we have $\mathcal{T}(G_{\alpha+1})$ and $\mathcal{S}_{\alpha+1}$ that satisfy the Theorem for $\beta-1$, with $C_{\alpha+1} = 3^{n-u}\left(\frac{2}{3}\right)^{m-u}\left(\frac{3}{4}\right)^{\beta-2}$ full-paths, each of which represents a proper coloring of $G_{\alpha+1}$ such that the color of v_1 is 1 and $\mathcal{S}_{\alpha+1}$ is the sub-collection of these paths of size at least 3/4 of $C_{\alpha+1}$, where each color class at level α other than $P_{\alpha+1}$ holds an equal number of colorings, the size of which is at least 2/9 of the colorings in $\mathcal{S}_{\alpha+1}$.

STEP 1 *$\mathcal{T}(U_\alpha)$ is the tree with height is u and is colored (as previously described) so that the color of the children are distinct and do not equal that of the parent. For simplicity, assume the color of the root node is 1. Then 1 is the* primary *color of the interval, while $\{2,3,4\}$ are the set of* secondary *colors.*

There are 3^{u-1} full paths in $\mathcal{T}(U_\alpha)$.

STEP 2 *$\mathcal{T}^*(U_\alpha)$ is formed by removing (if necessary) from $\mathcal{T}(U_\alpha)$ the interval-paths that use more than three colors. As was shown in Lemma 2.3.2, $\mathcal{T}^*(U_\alpha)$ has*

$3(2^{u-1}-1)$ *unique interval-paths for* $u>1$. *For* $u=1$, $\mathcal{T}^*(U_\alpha)$ *is the single node colored by the primary color 1.*

There are $\mathcal{X} = \begin{cases} 1 & \text{for } u=1 \\ 3(2^{u-1}-1) & \text{for } u>1 \end{cases}$ full paths in $\mathcal{T}^*(U_\alpha)$.

STEP 3 *Form* $\mathcal{T}'(G_{\alpha+1})$ *from* $\mathcal{T}(G_{\alpha+1})$ *by keeping* $\mathcal{S}_{\alpha+1}$ *and removing all other full paths. Hang isomorphic copies of* $\mathcal{T}'(G_{\alpha+1})$ *off of the leaves of* $\mathcal{T}^*(U_\alpha)$ *by transposing the color of the root of* $\mathcal{T}'(G_{\alpha+1})$ *to match the color of the corresponding leaf in* $\mathcal{T}^*(U_\alpha)$.

The resulting tree has $\mathcal{X}|\mathcal{S}_{\alpha+1}| = \mathcal{X}\cdot\frac{3}{4}\cdot 3^{n-u}\left(\frac{2}{3}\right)^{m-u}\left(\frac{3}{4}\right)^{\beta-2}$ full paths.

STEP 4 *For each node* x *at level* α. *Following the path from* x *back to the root node, we encounter* $x_1, x_2, \ldots, x_u$ *at levels 1 through* u. *Remove* x *(and branch below it) if and only if the color of* x *is the same color as any one of the nodes* $x_1, x_2, \ldots, x_u$. *(The ones kept are called* good *colorings.) When finished, we are left with* $\mathcal{T}(G_\alpha)$ *whose full paths represent proper colorings of the coil* $G_\alpha = G$.

We calculate the number C of full paths in $\mathcal{T}(G_\alpha)$, each representing a proper coloring of $G_\alpha = G$ after STEP 4 by considering three cases.

Case 1 $u=1$

$P_\alpha = P_{\alpha+1} = 1$ and U_α is an interval of type $\overline{234}$. We see that 3(2/9)=2/3 are good colorings. So, there are

$$1\cdot\frac{3}{4}\cdot 3^{n-1}\left(\frac{2}{3}\right)^{m-1}\left(\frac{3}{4}\right)^{\beta-2}\frac{2}{3} = 3^{n-1}\left(\frac{2}{3}\right)^{m}\left(\frac{3}{4}\right)^{\beta-1}$$

colorings, as desired.

Case 2 $u=2$

$P_\alpha = 1$, $\mathcal{X} = 3$ and the intervals are of type $\overline{34}$ ($\phi = 2$), $\overline{24}$ ($\phi = 3$), and $\overline{23}$ ($\phi = 4$). The hanging trees are transpositions of $\mathcal{T}(G_{\alpha+1})$, so that in the tree where $\phi = 2$, the color classes 1,3 and 4 each hold at least 2/9 of the colorings at level α (see the induction hypothesis). We are keeping the colors 3 and 4, that is, we are keeping $2 \cdot \frac{2}{9} = \frac{4}{9} = (\frac{2}{3})^2$ of the colorings. The argument is similar for $\phi = 3$ and $\phi = 4$, leaving

$$3 \cdot \frac{3}{4} \cdot \left(3^{n-2}\left(\frac{2}{3}\right)^{m-2}\left(\frac{3}{4}\right)^{\beta-2}\right)\left(\frac{2}{3}\right)^2 = 3^{n-1}\left(\frac{2}{3}\right)^m\left(\frac{3}{4}\right)^{\beta-1}$$

colorings, as desired.

Case 3 $u \geq 3$

Three of the $\mathcal{X} = 3(2^{u-1} - 1)$ intervals use exactly two colors, one of each of the forms $\overline{34}$, $\overline{24}$, and $\overline{23}$. By selection on level α below this intervals, we see that 2(2/9)=4/9 of the colorings are good. The remaining use exactly three colors and are evenly distributed among $\overline{2}$, $\overline{3}$, and $\overline{4}$. Below these such intervals we keep 2/9 of the colorings. We are keeping

$$\frac{3\left(\frac{4}{9}\right) + \left(3(2^{u-1}-1) - 3\right)\left(\frac{2}{9}\right)}{3(2^{u-1}-1)} = \frac{\left(\frac{2}{3}\right)^u \cdot 3^{u-1}}{3(2^{u-1}-1)}$$

of the colorings, that is,

$$3(2^{u-1}-1) \cdot \frac{3}{4} \cdot \left(3^{n-u}\left(\frac{2}{3}\right)^{m-u}\left(\frac{3}{4}\right)^{\beta-2}\right)\frac{\left(\frac{2}{3}\right)^u \cdot 3^{u-1}}{3(2^{u-1}-1)} = 3^{n-1}\left(\frac{2}{3}\right)^m\left(\frac{3}{4}\right)^{\beta-1}$$

colorings, as desired.

It remains to show: there exists a sub-collection $\mathcal{S}$ of full-paths in $\mathcal{T}(G_\alpha)$ of size at least 3/4 of C, where each color class at level $\alpha - 1$ other than 1 holds an equal number of colorings, the size of which is at least 2/9 of $|\mathcal{S}|$.

Notice that after STEP 4, a node x at level $\alpha - 1$ may have 0,1,2, or 3 children remaining in $\mathcal{T}(G_\alpha)$ depending on the number of its children that were trimmed. Consider all of the leaves of color s in $\mathcal{T}^*(U_\alpha)$. In STEP 3, the trees $\mathcal{T}(G_{\alpha+1})$ that are hung from those leaves, are exactly the same. So each interval-path of type $\overline{s}$ in $\mathcal{T}(U_\alpha)$ accounts for the same number of each of the colors in level $\alpha - 1$ of $\mathcal{T}(G_\alpha)$. Thus, the distribution of colors at level $\alpha - 1$ depends upon ϕ, the color of the leaf node at level u and the type of interval-path that extends from the root to that node.

Consider the following tables.

$\phi \setminus \overline{s}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
1	(Y,0,X,X)	(Y,X,0,X)	(Y,X,X,0)
2	–	(X,Y,0,X)	(X,Y,X,0)
3	(X,0,Y,X)	–	(X,X,Y,0)
4	(X,0,X,Y)	(X,X,0,Y)	–

Table 2. Distribution of next level upon trimming.

The sequences (C_1, C_2, C_3, C_4) in Table 2 show the distribution of the colors in row $\alpha - 1$ of $\mathcal{T}(G_\alpha)$, where ϕ is the color of the leaf of $\mathcal{T}^*(U_\alpha)$ and $\overline{s}$ is the type of interval-path that extends from the root to the given leaf. So that C_s is the number of surviving colorings with color s at level $\alpha - 1$ - counting multiple children of a node at level $\alpha - 1$.

Case 1 $u = 2$

There are three interval paths, all starting with the primary color 1 ending in one of each $\phi = 2, 3$, and 4. These paths are colored $\overline{34}$, $\overline{24}$, and $\overline{23}$ respectively. From $\phi = 2$, colored $\overline{34}$: $\overline{3}$ yields (X,Y,0,X) and $\overline{4}$ yields (X,Y,X,0). From $\phi = 3$, colored $\overline{24}$: $\overline{2}$ yields (X,0,Y,X) and $\overline{4}$ yields (X,X,Y,0). From $\phi = 4$, colored $\overline{23}$: $\overline{2}$ yields (X,0,X,Y) and $\overline{3}$ yields X,X,0,Y). This leaves a weighted distribution of (6X,2(X+Y),2(X+Y),2(X+Y)) at the $\alpha - 1$ level.

Again, we apply Lemma 2.3.7. In the case that $Y = 0$, $R = 0$. Since 6X $+\frac{R}{3} =$ 6X + 0 = 6X+ 6Y, we are done. Assume Y$\geq$ 1. Since 6X $+\frac{R}{3} \leq$ 6X + 1 $\leq$ 6X+ 6Y=3(2(X+Y)), we are done.

Case 2 $u \geq 3$

Case 2.1 *u is odd.*

There are $2^{u-1} - 1$ interval paths from level 1 through level u beginning with the primary color 1 and also ending in the primary color, that is, $\phi = 1$. Three of these are two-colored intervals. Considering the subtrees $\mathcal{T}(G_{\alpha+1})$ hanging from these vertices at level u, we use $\phi = 1$ and Table 1.

The path 1-2-1-... is $\overline{34}$: $\overline{3}$ yields (Y,X,0,X), $\overline{4}$ yields (Y,X,X,0).

The path 1-3-1-... is $\overline{24}$: $\overline{2}$ yields (Y,0,X,X), $\overline{4}$ yields (Y,X,X,0).

The path 1-4-1-... is $\overline{23}$: $\overline{2}$ yields (Y,0,X,X), $\overline{3}$ yields (Y,X,0,X).

These three intervals yield 2(3Y,2X,2X,2X).

Of the remaining intervals that end in 1, one-third are $\overline{2}$ each yielding (Y,0,X,X), one-third are $\overline{3}$ each yielding (Y,X,0,X), and one-third are $\overline{4}$ each yielding (Y,X,X,0). These interval paths yield $\frac{2^{u-1}-4}{3}$(3Y,2X,2X,2X).

Thus, from the trees hanging beneath these interval paths ending in the primary color 1, we keep a combined weighted distribution of $\frac{2^{u-1}+2}{3}$(3Y,2X,2X,2X).

There are $2^u - 2$ interval paths beginning with the primary color 1, ending in a secondary color, 2,3, or 4, that is $\phi = 2, 3,$ or 4. Due to symmetry, in one-third of these paths, $\phi = 2$. Also due to symmetry, half are $\overline{3}$ each yielding (X,Y,0,X), half are $\overline{4}$ each yielding (X,Y,X,0). In one-third of these paths, $\phi = 3$: half are $\overline{2}$ each yielding (X,0,Y,X), half are $\overline{4}$ each yielding (X,X,Y,0). In one-third of these paths, $\phi = 4$: half are $\overline{2}$ each yielding (X,0,X,Y), half are $\overline{3}$ each yielding (X,X,0,Y). These interval paths yield $\frac{2^{u-1}-1}{3}$(6X,2(X+Y),2(X+Y),2(X+Y)).

This leaves a total weighted distribution of $\frac{2^{u-1}-1}{3}$(6X+3Y,4X+2Y, 4X+2Y, 4X+2Y) + (3Y,2X,2X,2X) at the $\alpha - 1$ level.

Here, we let $A = \frac{2^{u-1}-1}{3}$(6X+3Y) + 3Y and $B = \frac{2^{u-1}-1}{3}$(4X+2Y) + 2X. Applying Lemma 2.3.7, and noticing that B is divisible by 2, we need only show that $A + \frac{R}{3} \leq 3B$ which is equivalent to showing

$$\frac{R}{3} \leq (2^{u-1} - 1)(2X) + (2^{u-1} - 4)Y + 6X$$

which is easily verified.

Case 2.2 *u is even.*

There are $2^{u-1} - 2$ interval paths beginning with the primary color 1 and also ending in the primary color 1, that is, $\phi = 1$. One-third are $\overline{2}$ each yielding (Y,0,X,X), one-third $\overline{3}$ each yielding (Y,X,0,X), and one-third $\overline{4}$ each yielding (Y,X,X,0). Combined, these yield $\frac{2^{u-1}-2}{3}$(3Y,2X,2X,2X).

There are $2^u - 1$ interval paths beginning with the primary color 1, ending in a secondary color 2, 3 or 4, that is, $\phi = 2, 3,$ or 4. Three of these are two-colored intervals.

The path 1-2-1-2-... in which $\phi = 2$ is $\overline{34}$: $\overline{3}$ yields (X,Y,0,X), $\overline{4}$ yields (X,Y,X,0).

The path 1-3-1-3-... in which $\phi = 3$ is $\overline{24}$: $\overline{2}$ yields (X,0,Y,X), $\overline{4}$ yields (X,X,Y,0).

The path 1-4-1-4-... in which $\phi = 4$ is $\overline{23}$: $\overline{2}$ yields (X,0,X,Y), $\overline{3}$ yields (X,X,0,Y).

These three intervals yield (6X,2(X+Y),2(X+Y),2(X+Y)).

Of the remaining intervals that end in a secondary number, one-third end in each 2,3, and 4. In one-third of these paths, $\phi = 2$: half are $\overline{3}$ each yielding (X,Y,0,X), half are $\overline{4}$ each yielding (X,Y,X,0). In one-third of these paths, $\phi = 3$: half are $\overline{2}$

each yielding (X,0,Y,X), half are $\overline{4}$ each yielding (X,X,Y,0). In one-third of these paths, $\phi = 4$: half are $\overline{2}$ each yielding (X,0,X,Y), half are $\overline{3}$ each yielding (X,X,0,Y). Combined, these yield $\frac{2^{u-1}-2}{3}$(6X,2(X+Y),2(X+Y),2(X+Y)).

From the trees hanging beneath these interval paths ending in a secondary number, we keep a combined weighted distribution of $\frac{2^{u-1}+1}{3}$(6X,2(X+Y),2(X+Y),2(X+Y)).

This leaves a total weighting of $\frac{2^{u-1}-2}{3}$(6X+3Y,4X+2Y, 4X+2Y, 4X+2Y) + (6X,2(X+Y),2(X+Y),2(X+Y)) at the $\alpha - 1$ level.

Applying Lemma 2.3.7, we need only show that $P + \frac{R}{3} \leq 3S$ which is equivalent to showing

$$\frac{R}{3} \leq (2^{u-1} - 2)(2X) + (2^{u-1} - 2)Y + 6Y$$

which is easily verified.

Case 3 $u = 1$

There is only one interval path, the single node colored 1 and $\phi = 1$. This interval is of the form $\overline{234}$: $\overline{2}$ yields (Y,0,X,X), $\overline{3}$ yields (Y,X,0,X), and $\overline{4}$ yields (Y,X,X,0). This leaves a weighted distribution of (3Y,2X,2X,2X) at the $\alpha - 1$ level. Setting A =3Y and B =2X and applying Lemma 2.3.7, we need only show 3Y + $\frac{R}{3} \leq$ 6X. (Note that $r = 0$ since the secondary class is even.)

We need more information about the distribution (3Y, 2X, 2X, 2X).

Case 3.1 *$upd(v_{\alpha+1}) > 1$.*

Let $upd(v_{\alpha+1}) = p$

Case 3.1.1 *p is odd.*

There are $2^{p-1} - 1$ interval paths from level 1 through level p beginning with the primary color 1 and also ending in the primary color, that is, $\phi = 1$. Three of these are two-colored intervals: $\overline{34}$, $\overline{24}$, and $\overline{23}$. Of the remaining $2^{p-1}-4$ interval paths, they are equally distributed $\overline{2}$, $\overline{3}$, and $\overline{4}$. Since we are keeping 2's, 3's, and 4's, at level α, we look at what we are leaving at level $\alpha-1$ for each of the 6 specific situations as described in Table 3 , $\phi=1$.

$\phi=1$ $U_{\alpha+1} \setminus U_\alpha$	$\overline{2}$	$\overline{3}$	$\overline{4}$
2	–	(y,x,0,w)	(y,x,w,0)
3	(y,0,x,w)	–	(y,x,w,0)
4	(y,0,w,x)	(y,w,0,x)	–
$\phi=2$ $U_{\alpha+1} \setminus U_\alpha$	$\overline{2}$	$\overline{3}$	$\overline{4}$
2	–	(h,g,0,h)	(h,g,h,0)
3	(z,0,v,z)	–	(w,y,x,0)
4	(z,0,z,v)	(w,y,0,x)	–
$\phi=3$ $U_{\alpha+1} \setminus U_\alpha$	$\overline{2}$	$\overline{3}$	$\overline{4}$
2	–	(z,v,0,z)	(w,x,y,0)
3	(h,0,g,h)	–	(h,h,g,0)
4	(w,0,y,x)	(z,z,0,v)	–
$\phi=4$ $U_{\alpha+1} \setminus U_\alpha$	$\overline{2}$	$\overline{3}$	$\overline{4}$
2	–	(w,x,0,y)	(z,v,z,0)
3	(w,0,x,y)	–	(z,v,z,0)
4	(g,0,g,h)	(g,g,0,h)	–

Table 3. Distribution of next level upon trimming when updegree is 1.

The single path 1-2-1-... is $\overline{34}$, so at level $\alpha+1$, we kept 3's and 4's. From the 3's at level $\alpha+1$, we are keeping 2's and 4's at level α. From the 3-2's, we keep $(y,0,x,w)$. From the 3-4's, we keep $(y,x,w,0)$. From the 4's at level $\alpha+1$, we are keeping 2's and 3's at level α. From the 4-2's, we keep $(y,0,w,x)$. From the 4-3's, we keep $(y,w,0,x)$.

The single path 1-3-1-... is $\overline{24}$, so at level $\alpha+1$, we kept 2's and 4's. From the 2's at level $\alpha+1$, we are keeping 3's and 4's at level α. From the 2-3's, we keep

$(y, x, 0, w)$. From the 2-4's, we keep $(y, x, w, 0)$. From the 4's at level $\alpha + 1$, we are keeping 2's and 3's at level α. From the 4-2's, we keep $(y, 0, w, x)$. From the 4-3's, we keep $(y, w, 0, x)$.

The single path 1-4-1-... is $\overline{23}$, so at level $\alpha + 1$, we kept 2's and 3's. From the 2's at level $\alpha + 1$, we are keeping 3's and 4's at level α. From the 2-3's, we keep $(y, x, 0, w)$. From the 2-4's, we keep $(y, x, w, 0)$. From the 3's at level $\alpha + 1$, we are keeping 2's and 4's at level α. From the 3-2's, we keep $(y, 0, x, w)$. From the 3-4's, we keep $(y, x, w, 0)$.

These three interval-paths yield $2(6y, 2x + 2w, 2x + 2w, 2x + 2w)$.

Of the $2^{p-1} - 4$ interval-paths that end in 1, one-third are $\overline{2}$, so we kept 2's. From these 2's at level $\alpha + 1$, we are keeping 3's and 4's at level α. From the 2-3's, we keep $(y, x, 0, w)$. From the 2-4's, we keep $(y, x, w, 0)$. These three interval-paths yield $\frac{2^{p-1}-4}{3}$ $(2y, 2x, w, w)$.

Of the $2^{p-1} - 4$ interval-paths that end in 1, one-third are $\overline{3}$, so we kept 3's. From these 3's at level $\alpha + 1$, we are keeping 2's and 4's at level α. From the 3-2's, we keep $(y, 0, x, w)$. From the 3-4's, we keep $(y, x, w, 0)$. These three interval-paths yield $\frac{2^{p-1}-4}{3}$ $(2y, w, 2x, w)$.

Of the $2^{p-1} - 4$ interval-paths that end in 1, one-third are $\overline{4}$, so we kept 4's. From these 4's at level $\alpha + 1$, we are keeping 2's and 3's at level α. From the 4-2's, we keep $(y, 0, w, x)$. From the 4-3's, we keep $(y, w, 0, x)$. These three interval-paths yield $\frac{2^{p-1}-4}{3}$ $(2y, w, w, 2x)$.

The total distribution at $\alpha - 1$ from the $\phi = 1$ intervals is $\frac{2^{p-1}+2}{3}$ $(6y, 2x + 2w, 2x + 2w, 2x + 2w)$.

There are $\frac{2^p-2}{3}$ interval paths from level 1 through level p beginning with the primary color 1 and ending in the secondary color 2, that is, $\phi = 2$.

Of the $\frac{2^p-2}{3}$ that and in 2, half are $\overline{3}$, so we kept 3's. From these 3's at level $\alpha + 1$, we are keeping 2's and 4's at level α. From the 3-2's, we keep $(z, 0, v, z)$. From the 3-4's, we keep $(w, y, x, 0)$. These three interval-paths yield $\frac{2^p-2}{6}$ $(z+w, y, x+v, z)$. Of the $\frac{2^p-2}{3}$ that and in 2, half are $\overline{4}$, so we kept 4's. From these 4's at level $\alpha+1$, we are keeping 2's and 3's at level α. From the 4-2's, we keep $(z, 0, z, v)$. From the 4-3's, we keep $(w, y, 0, x)$. These three interval-paths yield $\frac{2^p-2}{6}$ $(z+w, y, z, x+v)$. The total distribution at $\alpha-1$ from the $\phi = 2$ intervals is $\frac{2^p-2}{6}$ $(2z+2w, 2y, x+v+z, x+z+v)$.

We calculate the distribution arising from $\phi = 3$ and $\phi = 4$ in the same manner (using Table 3) yielding $\frac{2^p-2}{6}$ $(2z+2w, x+v+z, 2y, x+z+v)$ and $\frac{2^p-2}{6}$ $(2z+2w, x+v+z, x+z+v, 2y)$ respectively.

The total combined distribution is

$\frac{2^{p-1}+2}{3}$ $(6y, 2x+2w, 2x+2w, 2x+2w)$ + $\frac{2^p-2}{6}$ $(2z+2w, 2y, x+v+z, x+z+v)$ + $\frac{2^p-2}{6}$ $(2z+2w, x+v+z, 2y, x+z+v)$ + $\frac{2^p-2}{6}$ $(2z+2w, x+v+z, x+z+v, 2y)$ = $\frac{2^{p-1}+2}{3}$ $(6y, 2x+2w, 2x+2w, 2x+2w)$ + $\frac{2^{p-1}-1}{3}$ $(6z+6w, 2y+2x+2v+2z, 2y+2x+2v+2z, 2y+2x+2v+2z)$.

Hence, there are $(2^{p-1})(2y+2z+2w)+6y$ 1's.

There are $\frac{2^{p-1}-1}{3}(2y+4x+2w+2v+2z)+(2x+2w)$ 2's, 3's and 4's.

For simplicity, let $C = 2^{p-1}-1$. So, we need to show $C(2y+2z+2w)+6y \leq C(2y+4x+2w+2v+2z)+6x+6w$. Noting that $C \geq 3$ for all odd $p > 1$, simple algebra shows this is the equivalent of showing $y \leq 3x+v+w$.

Hence, we need to show that $y \leq 3x+v$.

Case 3.1.2 *p is even.*

The argument is similar, yielding the same inequality.

Case 3.2 *$upd(V_{\alpha+1}) = 1$.*

We use the results from the previous case.

2.5 Future Work

The depth-first search tree of a planar graph breaks the graph into coils. There is no record of any attempt to four-color a coil, so this is a new unsolved problem. In our attempt to solve this problem, we discovered many propositions which have led us to our current state. We have a computer program that has validated the conjecture for workable values of n. We are looking to see the pattern of behavior for our last case. If we can isolate a pattern, we might understand better the inequality. So far, the computer generated patterns show us that the inequality that we need does hold, that is, our conjecture has not been disproven.

APPENDIX

```
N = N;
K = 4;
perms = nextperm(N,K);
for (ii = 1:((prod (1:N)/(prod(1:(N-K))))))

P(ii,:,:) = perms();
PERMS(ii,:) = squeeze (P(ii,:,:))';

end

L = N;
O = 2;
perms = nextperm(L,O);
for (ii = 1:((prod (1:L)/(prod(1:(L-O))))))
T(ii,:,:) = perms();
TABLE(ii,:) = squeeze (T(ii,:,:))';
end

for (jj = 1:((prod (1:N)/(prod(1:(N-K))))))
for (kk = 1:((prod (1:L)/(prod(1:(L-O))))))

if (PERMS(jj,1:2) == TABLE(kk,:))
 P3(jj,1) = ceil(kk/2);
end

if (PERMS(jj,2:3) == TABLE(kk,:))
 P3(jj,2) = ceil(kk/2);
end

if (PERMS(jj,3:4) == TABLE(kk,:))
 P3(jj,3) = ceil(kk/2);
end

end
end
%%
SOL = 0;
% First 0
for (a = 1:((prod (1:N)/(prod(1:(N-K))))))
if (isempty(intersect(P3(1,:),P3(a,:))))
```

```
% First 1
for (b = 1:((prod (1:N)/(prod(1:(N-K))))))
if (~isempty(intersect(P3(b,:),P3(1,:)))
&& length(intersect(P3(b,:),P3(1,:)))<3)
if (~isempty(intersect(P3(b,:),P3(a,:)))
&& length(intersect(P3(b,:),P3(a,:)))<3)

% Second 0
for (c = 1:((prod (1:N)/(prod(1:(N-K))))))
if (isempty(intersect(P3(b,:),P3(c,:))))
if (~isempty(intersect(P3(c,:),P3(1,:)))
&& length(intersect(P3(c,:),P3(1,:)))<3)
if (~isempty(intersect(P3(c,:),P3(a,:)))
&& length(intersect(P3(c,:),P3(a,:)))<3)

% Second 1
for (d = b+1:((prod (1:N)/(prod(1:(N-K))))))
if (~isempty(intersect(P3(d,:),P3(1,:)))
&& length(intersect(P3(d,:),P3(1,:)))<3)
if (~isempty(intersect(P3(d,:),P3(a,:)))
&& length(intersect(P3(d,:),P3(a,:)))<3)
if (~isempty(intersect(P3(d,:),P3(b,:)))
&& length(intersect(P3(d,:),P3(b,:)))<3)
if (~isempty(intersect(P3(d,:),P3(c,:)))
&& length(intersect(P3(d,:),P3(c,:)))<3)

% Third 0
for (e = 1:((prod (1:N)/(prod(1:(N-K))))))
if (isempty(intersect(P3(e,:),P3(d,:))))
if (~isempty(intersect(P3(e,:),P3(c,:)))
&& length(intersect(P3(e,:),P3(c,:)))<3)
if (~isempty(intersect(P3(e,:),P3(1,:)))
&& length(intersect(P3(e,:),P3(1,:)))<3)
if (~isempty(intersect(P3(e,:),P3(a,:)))
&& length(intersect(P3(e,:),P3(a,:)))<3)
if (~isempty(intersect(P3(e,:),P3(b,:)))
&& length(intersect(P3(e,:),P3(b,:)))<3)

% Third 1
for (f = d+1:((prod (1:N)/(prod(1:(N-K))))))
if (~isempty(intersect(P3(f,:),P3(1,:)))
&& length(intersect(P3(f,:),P3(1,:)))<3)
if (~isempty(intersect(P3(f,:),P3(a,:)))
&& length(intersect(P3(f,:),P3(a,:)))<3)
```

```
if (~isempty(intersect(P3(f,:),P3(b,:)))
&& length(intersect(P3(f,:),P3(b,:)))<3)
if (~isempty(intersect(P3(f,:),P3(c,:)))
&& length(intersect(P3(f,:),P3(c,:)))<3)
if (~isempty(intersect(P3(f,:),P3(d,:)))
&& length(intersect(P3(f,:),P3(d,:)))<3)
if (~isempty(intersect(P3(f,:),P3(e,:)))
&& length(intersect(P3(f,:),P3(e,:)))<3)

% Fourth 0
for (g = 1:((prod (1:N)/(prod(1:(N-K))))))
if (isempty(intersect(P3(g,:),P3(f,:))))
if (~isempty(intersect(P3(g,:),P3(1,:)))
&& length(intersect(P3(g,:),P3(1,:)))<3)
if (~isempty(intersect(P3(g,:),P3(a,:)))
&& length(intersect(P3(g,:),P3(a,:)))<3)
if (~isempty(intersect(P3(g,:),P3(b,:)))
&& length(intersect(P3(g,:),P3(b,:)))<3)
if (~isempty(intersect(P3(g,:),P3(c,:)))
&& length(intersect(P3(g,:),P3(c,:)))<3)
if (~isempty(intersect(P3(g,:),P3(d,:)))
&& length(intersect(P3(g,:),P3(d,:)))<3)
if (~isempty(intersect(P3(g,:),P3(e,:)))
&& length(intersect(P3(g,:),P3(e,:)))<3)

% Fourth 1
for (h = f+1:((prod (1:N)/(prod(1:(N-K))))))
if (~isempty(intersect(P3(h,:),P3(1,:)))
&& length(intersect(P3(h,:),P3(1,:)))<3)
if (~isempty(intersect(P3(h,:),P3(a,:)))
&& length(intersect(P3(h,:),P3(a,:)))<3)
if (~isempty(intersect(P3(h,:),P3(b,:)))
&& length(intersect(P3(h,:),P3(b,:)))<3)
if (~isempty(intersect(P3(h,:),P3(c,:)))
&& length(intersect(P3(h,:),P3(c,:)))<3)
if (~isempty(intersect(P3(h,:),P3(d,:)))
&& length(intersect(P3(h,:),P3(d,:)))<3)
if (~isempty(intersect(P3(h,:),P3(e,:)))
&& length(intersect(P3(h,:),P3(e,:)))<3)
if (~isempty(intersect(P3(h,:),P3(f,:)))
&& length(intersect(P3(h,:),P3(f,:)))<3)
if (~isempty(intersect(P3(h,:),P3(g,:)))
&& length(intersect(P3(h,:),P3(g,:)))<3)
```

```
% Fifth 0
for (i = 1:((prod (1:N)/(prod(1:(N-K))))))
if (isempty(intersect(P3(i,:),P3(h,:))))
if (~isempty(intersect(P3(i,:),P3(1,:)))
&& length(intersect(P3(i,:),P3(1,:)))<3)
if (~isempty(intersect(P3(i,:),P3(a,:)))
&& length(intersect(P3(i,:),P3(a,:)))<3)
if (~isempty(intersect(P3(i,:),P3(b,:)))
&& length(intersect(P3(i,:),P3(b,:)))<3)
if (~isempty(intersect(P3(i,:),P3(c,:)))
&& length(intersect(P3(i,:),P3(c,:)))<3)
if (~isempty(intersect(P3(i,:),P3(d,:)))
&& length(intersect(P3(i,:),P3(d,:)))<3)
if (~isempty(intersect(P3(i,:),P3(e,:)))
&& length(intersect(P3(i,:),P3(e,:)))<3)
if (~isempty(intersect(P3(i,:),P3(f,:)))
&& length(intersect(P3(i,:),P3(f,:)))<3)
if (~isempty(intersect(P3(i,:),P3(g,:)))
&& length(intersect(P3(i,:),P3(g,:)))<3)

% Fifth 1
for (j = h+1:((prod (1:N)/(prod(1:(N-K))))))
if (~isempty(intersect(P3(j,:),P3(1,:)))
&& length(intersect(P3(j,:),P3(1,:)))<3)
if (~isempty(intersect(P3(j,:),P3(a,:)))
&& length(intersect(P3(j,:),P3(a,:)))<3)
if (~isempty(intersect(P3(j,:),P3(b,:)))
&& length(intersect(P3(j,:),P3(b,:)))<3)
if (~isempty(intersect(P3(j,:),P3(c,:)))
&& length(intersect(P3(j,:),P3(c,:)))<3)
if (~isempty(intersect(P3(j,:),P3(d,:)))
&& length(intersect(P3(j,:),P3(d,:)))<3)
if (~isempty(intersect(P3(j,:),P3(e,:)))
&& length(intersect(P3(j,:),P3(e,:)))<3)
 if (~isempty(intersect(P3(j,:),P3(f,:)))
&& length(intersect(P3(j,:),P3(f,:)))<3)
if (~isempty(intersect(P3(j,:),P3(g,:)))
&& length(intersect(P3(j,:),P3(g,:)))<3)
if (~isempty(intersect(P3(j,:),P3(h,:)))
&& length(intersect(P3(j,:),P3(h,:)))<3)
if (~isempty(intersect(P3(j,:),P3(i,:)))
&& length(intersect(P3(j,:),P3(i,:)))<3)

% Sixth 0
```

```
for (k = 1:((prod (1:N)/(prod(1:(N-K))))))
if (isempty(intersect(P3(k,:),P3(j,:))))
if (~isempty(intersect(P3(k,:),P3(1,:)))
&& length(intersect(P3(k,:),P3(1,:)))<3)
if (~isempty(intersect(P3(k,:),P3(a,:)))
&& length(intersect(P3(k,:),P3(a,:)))<3)
if (~isempty(intersect(P3(k,:),P3(b,:)))
&& length(intersect(P3(k,:),P3(b,:)))<3)
if (~isempty(intersect(P3(k,:),P3(c,:)))
&& length(intersect(P3(k,:),P3(c,:)))<3)
if (~isempty(intersect(P3(k,:),P3(d,:)))
&& length(intersect(P3(k,:),P3(d,:)))<3)
if (~isempty(intersect(P3(k,:),P3(e,:)))
&& length(intersect(P3(k,:),P3(e,:)))<3)
if (~isempty(intersect(P3(k,:),P3(f,:)))
&& length(intersect(P3(k,:),P3(f,:)))<3)
if (~isempty(intersect(P3(k,:),P3(g,:)))
&& length(intersect(P3(k,:),P3(g,:)))<3)
if (~isempty(intersect(P3(k,:),P3(h,:)))
&& length(intersect(P3(k,:),P3(h,:)))<3)
if (~isempty(intersect(P3(k,:),P3(i,:)))
&& length(intersect(P3(k,:),P3(i,:)))<3)
SOL = SOL + 1;
disp('Solution number:');
disp(SOL);
disp([P3(a,:); P3(b,:); P3(c,:); P3(d,:); P3(e,:);
P3(f,:); P3(g,:); P3(h,:); P3(i,:); P3(j,:); P3(k,:)]);

end
end
end
end
end
end
end
end
end
end
end
end
% end Sixth 0
end
end
end
```

```
end
end
end
end
end
end
end
end
% end Fifth 1
end
end
end
end
end
end
end
end
end
end
% end Fifth 0
end
end
end
end
end
end
end
end
end
% end Fourth 1
end
end
end
end
end
end
end
end
% end Fourth 0
end
end
end
end
end
```

```
end
end
% end Third 1
end
end
end
end
end
end
% end Third 0
end
end
end
 end
end
% end Second 1
end
end
end
end
% end Second 0
end
end
end
% end First 1
end
end
```

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